Math 132: Differential Topology

§ 1-manifolds and some consequences

Thm (Classification of 1-manifolds)

Every compact, connected 1-manifold with boundary is diffeomorphic to S^1 or $I = [0,1]$.

(See the Appendix of [Milnor] or [Guillemin-Pollack] for the proof. The idea is to parametrize the 1-manifold by arc length, in a maximal way.)

Cor The boundary of any compact 1-manifold with boundary consists of an even number of points. ☆

As we will see, this seemingly trivial corollary has numerous nontrivial applications.

Later, in the next chapter, we will refine this statement by counting those boundary points with sign (orientation). Such signed count is always 0, and this fact is at the heart of oriented intersection theory.

2/ Let M be a compact manifold with boundary.

<u>Lemma</u> There's no smooth map $f: M \to \partial M$ that leaves ∂M pointwise fixed.

That is, there is no retraction of M onto ∂M.

proof) Suppose there were such a map f.

Let $y \in \partial M$ be a regular value for f (which exists thanks to Sard).

Since y is certainly a regular value for the identity map $f|_{\partial M}$ as well,

$f^{-1}(y)$ is a smooth 1-manifold, with boundary

$$\partial f^{-1}(y) = f^{-1}(y) \cap \partial M = \{y\}.$$

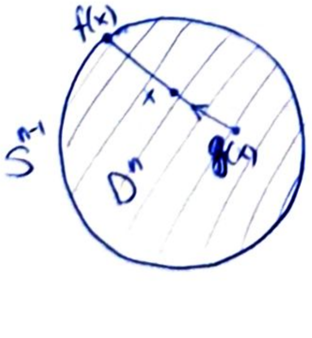
But $f^{-1}(y)$ is also compact, so that $\# |\partial f^{-1}(y)|$ must be even,

a contradiction. ∎

<u>Thm</u> (Smooth Brouwer fixed point theorem)

Any smooth map $g: D^n \to D^n$ has a fixed point.

proof) Suppose g has no fixed point. Then we can construct a retraction $f: D^n \to S^{n-1}$ which is impossible from the lemma:



$f(x) = x + tu$, where

$$u = \frac{x - g(x)}{\|x - g(x)\|} \quad \text{and} \quad t = -x \cdot u + \sqrt{1 - x \cdot x + (x \cdot u)^2}.$$

↑ strictly positive.

∎

3/

Often, one can use an approximation theorem to pass to the continuous case, and this theorem is an example:

<u>Thm</u> (Brouwer fixed point theorem)

Any continuous map $G : D^n \to D^n$ has a fixed point.

proof) Given $\varepsilon > 0$, thanks to the <u>Weierstrass approximation theorem</u>, there is a polynomial function $P_0 : \mathbb{R}^n \to \mathbb{R}^n$ with $\|P_0(x) - G(x)\| < \varepsilon$ for $x \in D^n$.

However, P_0 may send points of D^n into points outside of D^n, so to correct this we set $P(x) = \dfrac{P_0(x)}{1+\varepsilon}$.

Clearly, $P : D^n \to D^n$ and $\|P(x) - G(x)\| < \dfrac{2\varepsilon}{1+\varepsilon} < 2\varepsilon$ for $x \in D^n$.

Suppose G has no fixed points. Then the continuous function $\|G(x) - x\|$ must take on a minimum $\mu > 0$ on D^n. Choosing $P : D^n \to D^n$ as above, with $\|P(x) - G(x)\| < \mu$ for all $x \in D^n$, we have $P(x) \neq x$ for all $x \in D^n$.

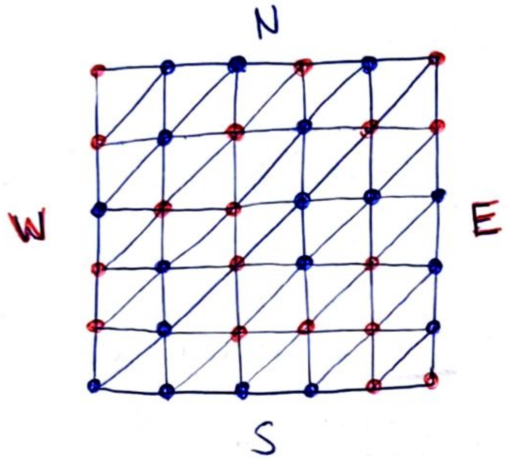
Since P is smooth, this contradicts the Smooth Brouwer fixed point theorem. ■

The following is just for fun:

(Hex theorem)

<u>Thm</u> There's no draw in the game of Hex. That is, if we color the vertices of a triangulation of I^2 in 2 colors, say red and blue, then either the red connects E–W or the blue connects N–S.

proof)



Suppose there is a coloring for which neither red nor blue connects the two opposite sides. Then, we can construct a continuous map $f: I^2 \to I^2$ without any fixed point, which contradicts Brouwer fixed point theorem.

We construct such f as follows: For each vertex v of the triangulation, set

$$f(v) = \begin{cases} v + (\varepsilon, 0) & \text{if } v \text{ is colored in red and connected to } W, \\ v - (\varepsilon, 0) & \text{if } v \text{ is colored in red and not connected to } W, \\ v + (0, \varepsilon) & \text{if } v \text{ is colored in blue and connected to } S, \\ v - (0, \varepsilon) & \text{if } v \text{ is colored in blue and not connected to } S. \end{cases}$$

Extend linearly on each simplex. Then f has no fixed points. ∎